Algebraic Geometry Lecture 24 – Toric Varieties

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Last week we went

$$\text{Toric variety} \quad \longrightarrow \quad \text{(Toric) fan.}$$

This week we go the other way.

Let our lattice be $N \cong \mathbb{Z}^2$. Consider the two-dimensional cone σ generated by $(0,1)$ and $(2,-1)$. Find the dual cone $\sigma^\vee \subset M = N^\vee = \mathrm{Hom}(N, \mathbb{Z})$. M has a basis of functionals h_1, h_2 that we'll write as $h_1 = (1,0)$ and $h_2 = (0,1)$. The dual cone is

$$\{m \in M \,:\, m(s) \geqslant 0 \text{ for every } s \in \sigma\}.$$

These are the "vectors in M at most orthogonal to everything in N". To see $\sigma^\vee$ it helps to tensor with $\mathbb{R}$.

Over $\mathbb{R}$, $\sigma^\vee \otimes \mathbb{R}$ is generated by $(1,2)$ and $(1,0)$. But $\sigma^\vee$ itself (which is really $M\cap$"the dual of σ") has three generators, $(1,0), (1,2)$, and $(1,1)$. This gives the semigroup associated to $\sigma^\vee$, called S_σ. (If it comes from a fan then it's finitely generated by Gordon's theorem.)

Write this group multiplicatively, so $(a,b)\cdot(c,d) = (a+c, b+d)$. Then we can form the semigroup algebra $\mathbb{C}S_\sigma$ whose elements are linear combinations of elements of S_σ with the induced multiplication.

⌜ Recall group algebras: Let G be a finite group, then elements of $\mathbb{C}G$ look like $\lambda_1 g_1 + \ldots + \lambda_n g_n$, and we multiply them by, for example,

$$(5g_1 + 2g_2)(\sqrt{-3}g_3) = 5\sqrt{-3}g_1g_3 + 2\sqrt{-3}g_2g_3.$$

Semigroup algebras work the same way. ⌟

If we write $X = (1,0), Y = (0,1)$, we see S_σ has generators

$$X, XY, XY^2.$$

So $\mathbb{C}S_\sigma = \mathbb{C}[X, XY, XY^2]$. We define a ring homomorphism

$$\mathbb{C}[u,v,w] \to \mathbb{C}S_\sigma$$

by

$$u \mapsto X \quad v \mapsto XY \quad w \mapsto XY^2.$$

This is clearly surjective and has kernel $\langle uw - v^2\rangle$, so by the first isomorphism theorem,

$$\mathbb{C}S_\sigma \cong \mathbb{C}[u,v,w]/(uw - v^2).$$

Finally to get our toric variety we let

$$U_\sigma = \operatorname{Spec} \mathbb{C}S_\sigma,$$

then we can see that $U_\sigma = \{(u,v,w) \in \mathbb{C}^3 \,:\, uw - v^2 = 0\}$.

[1]Typed by Lee Butler